Synopsis of the Ph.D. Thesis entitled

GENERALIZATION OF SECONDARY DOMINATION IN GRAPHS

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Synopsis

GENERALIZATION OF SECONDARY DOMINATION IN GRAPHS

Graph Theory is a main branch of Mathematics and it has applications in diverse fields which include Computer Science (Algorithms and computation), Biochemistry (Genomics), Electrical Engineering (Communication networks and Coding theory) and Operations Research (Scheduling). One of the main emerging concepts in Graph theory is Domination in graphs.

Domination is widely applied in facility location problems, where the number of facilities (e.g. hospitals, fire stations, etc.) is fixed and one attempts to minimize the distance that a person needs to travel to get to the closest facility. A similar problem occurs when the maximum distance to a facility is fixed and one attempts to minimize the number of facilities necessary so that every one is serviced. Concepts from domination also appear in problems involving finding sets of representatives, in monitoring communication or electrical networks and in land surveying (e.g. minimizing the number of places a surveyor must stand in order to take height measurements for an entire region).

Let $G = (V, E)$ be a simple graph. A set $S \subseteq V$ is a **dominating set** of G if every vertex in $V - S$ is adjacent to a vertex in S. The **domination number** of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set.

A thorough study of domination appears in [4, 5]. Many domination parameters are introduced by imposing additional constraints on the dominating set S or on the dominated set $V - S$ or on the method by which vertices in $V - S$ are dominated.

In our thesis we deal with two domination parameters :$(1, 2)$ - Domination and Total $(1, 2)$ - Domination. $(1, 2)$ - Domination was already introduced by Hedetniemi et al.[6] in 2008, whereas the Total $(1, 2)$ - domination is a new parameter introduced by us.

$(1, 2)$ - domination is defined by imposing a condition on the method by which vertices in $V - S$ are dominated.

A subset $S \subseteq V$ of vertices is called a $(\mathbf{1}, \mathbf{2})$ **- dominating set** of a graph G if for every vertex $v \in V - S$, there are two distinct vertices u, $w \in S$ such that u is adjacent to v and w is at a distance of at most 2 from v. The $(1, 2)$ - domination number of G is the minimum cardinality of a $(1, 2)$ - dominating set of G and it is denoted by $\gamma_{1,2}(G)$.

Given a dominating set $S \subseteq V$ in a graph $G = (V, E)$, place one guard at each vertex in S. Should there be a problem at a vertex $v \in V - S$, we can send a guard at a vertex $u \in S$ adjacent to v to handle the problem. If for some reason this guard needs assistance, a second guard can be sent from S to v, but the question is : how long will it take for a second guard to arrive? This is the issue of what we call **secondary domination** [6].

In 2008, Hedetniemi et al.[6] have introduced the concept of secondary domination. A subset $S \subseteq V$ of vertices is called a $(\mathbf{1}, \mathbf{k})$ **- dominating set** in G if for every vertex $v \in V - S$, there are two distinct vertices $u, w \in S$ such that u is adjacent to v, and w is at a distance of at most k from v. (Note that k is a positive integer). The $(\mathbf{1}, \mathbf{k})$ **- domination number**, $\gamma_{1,k}(G)$ is the minimum cardinality of a $(1, k)$ - dominating set.

A set $S \subseteq V$ is a **independent** $(\mathbf{1}, \mathbf{k})$ **- dominating set** if S is a $(1, k)$ - dominating set that is also an independent set.

Since $V(G)$ is a $(1, k)$ - dominating set of any graph G, $(1, k)$ - dominating set exists for all the graphs. But the existence of independent $(1, k)$ - dominating set is questionable: for example, K_n does not have an independent $(1, k)$ - dominating set. Hence the main results in [6] focus on identifying the classes of graphs, that have an independent $(1, 2)$ - dominating sets. The following results on $(1, k)$ - dominating sets deal with different types of dominating sets that are also $(1, k)$ - dominating sets, for $k \leq 4$ [6].

1. Every dominating set of cardinality at least 2 in a connected graph G with $\gamma(G) \geq 2$ is a $(1,4)$ - set.

2. No tree of order at least 3 has a γ - set or an i - set that is also a $(1,1)$ - set.

3. If D is a γ - set in a tree $T \neq K_{1,n}$ then D is either a $\gamma_{1,2}$ - set a $\gamma_{1,3}$ - set or a $\gamma_{1,4}$ - set.

4. Every maximal independent set in a connected graph G with $\gamma(G) \geq 2$ is a $(1,4)$ - set.

5. Every 2 - maximal independent set in a connected graph G with $\gamma(G) \geq 2$ is a $(1,3)$ - set.

Fink and Jacobson [2, 3] introduced the concept of k - dominating sets in general and in particular 2 - dominating sets in 1985.

A set S is a **2 - dominating set** if every vertex $v \in V - S$ is dominated by at least two vertices in S. The minimum cardinality of a 2 - dominating set is denoted by $\gamma_2(G)$.

A 2 - dominating set is same as a $(1,1)$ - set. Since any 2 - dominating set is also a $(1,2)$ - dominating set, $\gamma_{1,2}(G) \leq \gamma_{1,1}(G) = \gamma_2(G)$.

Total domination was introduced by Cockayne et al.[1] in 1980.

A set S is a **total dominating set** if for every vertex $v \in V$, there is a vertex $u(\neq v) \in S$, such that u is adjacent to v. The total domination number $\gamma_t(G)$ equals to the minimum cardinality of a total dominating set in G.

Note that every total dominating set is also a $(1,2)$ - dominating set. Therefore for every graph G, $\gamma_{1,2}(G) \leq \gamma_t(G)$.

Variation of $(1,2)$ - domination parameters, such as $(1,2)$ - double domination and $(1,2)$ - triple domination are studied in [7, 8].

A subset $S \subseteq V$ of vertices is called a **Total $(1,2)$ - dominating set** of a graph G if for every $v \in V$, there are two distinct vertices $u, w \in S$, such that u is adjacent to v and w is at a distance of at most 2 from v. The **total $(1,2)$ - domination number** is the minimum cardinality of a total $(1,2)$ - dominating set and it is denoted by $\gamma_{t(1,2)}$(G).

Summary of our Results

We present our results in **five chapters**.

Chapters 2, 3 and 4 deal with $(1,2)$ - domination. Chapters 5 and 6 deal with total $(1,2)$ - domination.

All our graphs are simple and undirected. Througout our thesis, n denotes the order of the graph.

Chapter 2 : $(1,2)$ **- Domination**

First, we characterize the minimal $(1,2)$ - dominating sets.

Theorem 1. A $(1,2)$ - dominating set S is a minimal $(1,2)$ - dominating set iff for each vertex v in S, one of the following conditions holds:

(i) v is an isolated vertex of S

(ii) there exists a vertex u in $V-S$ such that $N_2(u) \cap (S-\{v\})$ is a singleton set.

Next, we deduce the following observations that are cruical in determining $\gamma_{1,2}$ for standard graphs.

Observation 1. For every non-trivial graph G, with full - degree vertex, $\gamma_{1,2}(G) = 2$.

Observation 2. If G has a pendant vertex v, then every $\gamma_{1,2}$ - set of G contains either v or the support vertex of v.

Observation 3. If a support vertex is adjacent to more than one pendant vertex, then the support vertex must belong to every $\gamma_{1,2}$ - set.

Observation 4. For any two graphs G and H, $\gamma_{1,2}(G \cup H) = \gamma_{1,2}(G) + \gamma_{1,2}(H)$.

Next, we determine $\gamma_{1,2}$ for standard graphs such as complete graphs, cycles, paths, wheels, fans, complete bipartite graphs, double stars, wounded spiders and spiders.

The next two results deal with composition of graphs.

Theorem 2. Let G be a non-trivial connected graph. Then for any graph H, $\gamma_{1,2}(GoH) = |V(G)|$.

Corollary 1. Let G be any graph having t isolates. Then for any graph H, $\gamma_{1,2}(GoH) = |V(G)| + t$, where $t \geq 0$.

Our next result is the characterization of connected graphs with $\gamma_{1,2} = 2$. Using this result, we have characterized the trees with $\gamma_{1,2} = 2$.

Theorem 3. Let G be a connected graph. Then $\gamma_{1,2}(G) = 2$ iff $diam(G) \leq 3$ and one of the following conditions holds:

(i) a star or a double star is a spanning tree of G.

(ii) There exist two vertices u and v in $V(G)$ such that $N[u] \cup N[v] = V(G)$ and every vertex in $N(u) - N(v)$ is adjacent to at least one vertex in $N(v)$ and vice versa.

Corollary 2. Let T be a tree. Then $\gamma_{1,2}(T) = 2$ iff T is a star or a double star.

Next, we charaterize the graphs with $\gamma_{1,2} = n, n-1$.

Theorem 4. Let G be a non-trivial connected graph. Then $\gamma_{1,2}(G) = n$ iff $n = 2$ i.e. $\gamma_{1,2}(G) = n$ iff $G = K_2$.

Theorem 5. Let G be a connected graph. Then $\gamma_{1,2}(G) = n - 1$ iff $n = 3$ i.e. $\gamma_{1,2}(G) = n - 1$ iff $G = P_3$ *or* K_3.

Corollary 3. Let G be any graph. Then

(i) $\gamma_{1,2}(G) = n$ iff $G = sK_1 \cup kK_2$, where $s = n - 2k$ and $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

(ii) $\gamma_{1,2}(G) = n - 1$ iff $G = sK_1 \cup kK_2 \cup H$, where $H = P_3$ *or* K_3, $s \geq 0$ and $0 \leq k \leq \lfloor \frac{n-3}{2} \rfloor$.

Chapter 3 : Bounds of $\gamma_{1,2}$

First, we observe that, for any non-trivial graph G, $\gamma_{1,2}(G) \geq max\{2, \left\lceil \frac{n}{1+\Delta(G)} \right\rceil\}$.

Next, we derive bounds of $\gamma_{1,2}$ in terms of n and Δ.

Theorem 1. If G is a graph of order $n \geq 3$ with $\Delta(G) \geq n - 2$, then

$$\gamma_{1,2}(G) = \begin{cases} 2 \text{ if } G \textit{ is connected} \\ 3 \text{ if } G \textit{ is disconnected.} \end{cases}.$$

Theorem 2. Let G be a connected graph with $2 \leq \Delta(G) \leq n - 3$. Then $\gamma_{1,2}(G) \leq n - \Delta(G)$.

Next, we characterize the trees that achieve the upper bound.

Theorem 3. Let T be a tree with $n \geq 3$. Then $\gamma_{1,2}(T) = n - \Delta(T)$ iff T is a spider or a wounded spider or the broom graph $B(n, 4)$.

Theorem 4. For any tree T of order $n \geq 3$, $\gamma_{1,2}(T) \leq \lceil \frac{n}{2} \rceil$. Equality holds for spiders.

Using these bounds, we characterize the graphs with $\gamma_{1,2} = n-2, n-3$.

Theorem 5. Let G be a connected graph. Then $\gamma_{1,2}(G) = n-2$ iff $G = P_5$ or G is of order 4.

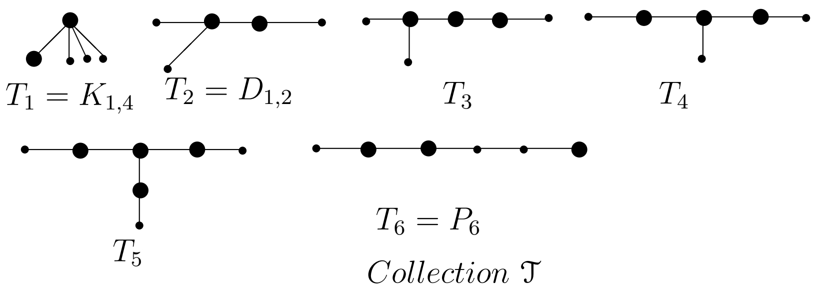
Corollary 1. $\gamma_{1,2}(G) = n-2$ iff G is one of the following graphs:

(i) $G = sK_1 \cup rK_2 \cup H$, where $H = 2P_3, 2K_3$ *or* $P_3 \cup K_3$, $s = n-2r-6$ and $0 \leq r \leq \lfloor \frac{n-6}{2} \rfloor$.

(ii) $G = sK_1 \cup P_5 \cup rK_2$, where $s = n-2r-5$ and $0 \leq r \leq \lfloor \frac{n-5}{2} \rfloor$.

(iii) $G = sK_1 \cup rK_2 \cup H$, where H is a connected graph of order 4, $s = n-2r-4$ and $0 \leq r \leq \lfloor \frac{n-4}{2} \rfloor$.

Theorem 6. Let T be a tree. Then $\gamma_{1,2}(T) = n-3$ iff $T \in \mathcal{T}$, where $\mathcal{T}$ is the collection of trees given in Fig.1.



Collection $\mathcal{T}$

Fig.1.

Theorem 7. Let G be a connected graph. Then $\gamma_{1,2}(G) = n-3$ iff $G \in \mathcal{T} \cup \mathcal{G}$, where $\mathcal{T}$ and $\mathcal{G}$ are the collections of graphs in Fig.1. and Fig.2.

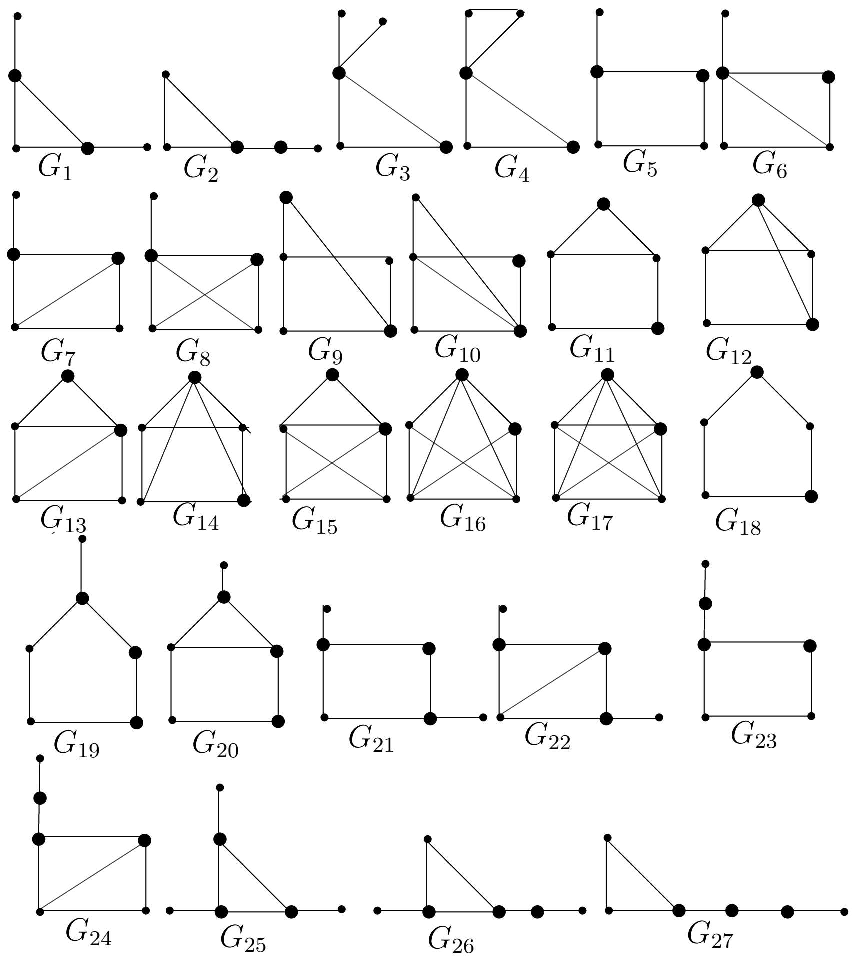
Corollary 2. $\gamma_{1,2}(G) = n-3$ iff G is one of the following graphs:

(i) $G = sK_1 \cup rK_2 \cup H$, where $H \in \mathcal{G} \cup \mathcal{T}$, $s = n-2r-|V(H)|$ and $0 \leq r \leq \left\lfloor \frac{n-|V(H)|}{2} \right\rfloor$.

(ii) $G = sK_1 \cup rK_2 \cup H_i \cup H_j$, where $H_i = P_3$ or K_3, H_j is of order 4, $s = n-2r-7$ and $0 \leq r \leq \lfloor \frac{n-7}{2} \rfloor$.

(iii) $G = sK_1 \cup rK_2 \cup H \cup P_5$, where $H = P_3$ or K_3, $s = n-2r-8$ and $0 \leq r \leq \lfloor \frac{n-8}{2} \rfloor$.

(iv) $G = sK_1 \cup rK_2 \cup H$, where $H = 2K_3 \cup P_3$, $2P_3 \cup K_3$, $3P_3$ or $3K_3$, $s = n-2r-9$ and $0 \leq r \leq \lfloor \frac{n-9}{2} \rfloor$.



Collection $\mathcal{G}$

*Fig.*2.

In the next two results, we consider the graphs with diameter 2.

Theorem 8. Let G be a connected graph with $2 \leq \Delta(G) \leq n-3$ and $diam(G) = 2$. Then $\gamma_{1,2}(G) \leq n - \Delta(G) - 1$. Equality holds for C_5.

Theorem 9. Let G be a graph with $diam(G) = 2$.

If $\delta(G) = 1$, then $\gamma_{1,2}(G) = 2$.

If $\delta(G) \geq 2$, then $\gamma_{1,2}(G) \leq \delta(G)$.

Chapter 4 : Bounds of $\gamma_{1,2}$ for Trees and Complement of Graphs

Throughout this chapter r denotes the number of pendant vertices.

In Chapter 3, we have proved that $\gamma_{1,2} \leq n - \Delta$. Now we have derived a stronger bound for trees in terms of the number of pendant vertices.

Theorem 1. Let T be a tree. Then

(i) $3 \leq \gamma_{1,2}(T) \leq n-r$, if $diam(T) \geq 5$

(ii) $\gamma_{1,2}(T) = n-r$, if $diam(T) = 4$.

Theorem 2. Let G be a connected graph with $diam(G) \geq 3$. Then $\gamma_{1,2}(G) \leq n - r_1$, where r_1 denotes the number of pendant vertices in G.

Next, we charaterize the lower bound of Theorem 1.

Theorem 3. Let T be a tree. Then $\gamma_{1,2}(T) = 3$ iff T is one of the following:

(i) T is obtained from P_5 by attaching zero or more pendant vertices to the internal vertices of P_5

(ii) T is obtained from P_6 by attaching zero or more pendant vertices to exactly one support vertex and its neighbour that is not a pendant vertex

(iii) $T = P_7$.

Next result deals with a necessary condition for a graph to have $\gamma_{1,2} = n-r$.

Theorem 4. Let T be a tree with $diam(T) \geq 3$, R be the set of all pendant vertices in T and $|R| = r$. If $\gamma_{1,2}(T) = n-r$, then one of the following holds:

(i) Every vertex in $V(T) - R$ is a support vertex

(ii) For every u in $V(T) - R$ that is not a support vertex, there exists at least one vertex (say) v_1 in $N(u)$ such that the vertices in $N(v_1) - \{u\}$ are pendant vertices in T.

Next, we deal with graph complements.

Theorem 5. For any graph G with $diam(G) \geq 3$, $\gamma_{1,2}(\overline{G}) = 2$.

Theorem 6. Let $G \neq K_n$ be a connected graph. Then

(i) $\gamma_{1,2}(\overline{G}) \leq \delta(G) + 2$ if $\overline{G}$ is disconnected

(ii) $\gamma_{1,2}(\overline{G}) \leq \delta(G) + 1$ if $\overline{G}$ is connected.

Next, we charaterize the connected graphs G with $\gamma_{1,2}(\overline{G}) = \delta(G) + 2$.

Theorem 7. Let G be a connected graph. Then $\gamma_{1,2}(\overline{G}) = \delta(G) + 2$ iff $\overline{G} = sK_1 \cup kK_2 \cup H$, where $s \geq 0$, $0 \leq k \leq \left\lfloor \frac{n-|V(H)|}{2} \right\rfloor$ and H has a full - degree vertex.

Next, we derive Nordhaus - Gaddum type bound.

Theorem 8. For any non-trivial graph G, $4 \leq \gamma_{1,2}(G) + \gamma_{1,2}(\overline{G}) \leq n + 2$.

Next, we characterize the trees with the lower bound.

Theorem 9. Let T be a non-trivial tree. Then $\gamma_{1,2}(T) + \gamma_{1,2}(\overline{T}) = 4$ iff $T = K_2$ or T is a double star.

Next, we characterize the graphs G with $\gamma_{1,2}(G) + \gamma_{1,2}(\overline{G}) = n + 1, n + 2$.

Theorem 10. Let G be a non-trivial connected graph. Then $\gamma_{1,2}(G) + \gamma_{1,2}(\overline{G}) = n + 2$ iff $G = K_n$ or $K_n - M$, where M is a matching of K_n.

Theorem 11. Let G be a disconnected graph. Then $\gamma_{1,2}(G) + \gamma_{1,2}(\overline{G}) = n + 2$ iff K_1 and K_2 are the only components of G. (i.e. $\gamma_{1,2}(G)+\gamma_{1,2}(\overline{G}) = n+2$ iff $G = sK_1 \cup kK_2$, $s = n-2k$ and $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$).

Theorem 12. Let G be a connected graph of order $n \geq 4$. Then $\gamma_{1,2}(G) + \gamma_{1,2}(\overline{G}) = n + 1$ iff $\overline{G} = sK_1 \cup kK_2 \cup H$, where $H = P_3$ *or* K_3, $s = n - 2k - 3$, $0 \leq k \leq \lfloor \frac{n-3}{2} \rfloor$ and $k + s \geq 1$.

Corollary 1. Let G be a disconnected graph of order $n \geq 4$. Then $\gamma_{1,2}(G) + \gamma_{1,2}(\overline{G}) = n + 1$ iff $G = sK_1 \cup kK_2 \cup H$, where $H = P_3$ *or* K_3, $s = n - 2k - 3$, $0 \leq k \leq \lfloor \frac{n-3}{2} \rfloor$ and $k + s \geq 1$.

Hedetniemi et al.[6] have determined the dominating chain $\gamma \leq \gamma_{1,2} \leq \gamma_t$. Next, we have proved that, $\gamma = \gamma_{1,2} = \gamma_t$ for some classes of graphs.

Theorem 13. If G is the complement of a tree $T \neq K_{1,n-1}$ with $n \geq 3$, then $\gamma(G) = \gamma_t(G) = \gamma_{1,2}(G) = 2$.

Theorem 14. If a graph G is the complement of a bipartite graph and G has no isolates, then $\gamma_{1,2}(G) = \gamma_t(G) = 2$.

Theorem 15. For the wounded spider $WS_{s,t}$, $\gamma = \gamma_t = \gamma_{1,2}$.

Theorem 16. For the spider, $\gamma_{1,2} = \gamma_t = \gamma_2$.

Chapter 5 : Total $(1,2)$ - domintion in graphs

In this chapter, we introduce a new domination parameter called **Total $(1,2)$ - domination** by imposing condition on the $(1,2)$ - dominating set.

Definition. A subset $S \subseteq V$ of vertices is called a **Total $(1,2)$ - dominating set** of a graph G if for every $v \in V$, there are two distinct vertices $u, w \in S$, such that u is adjacent to v and w is at a distance of at most 2 from v.

The **total $(1,2)$ - domination number** is the minimum cardinality of a total $(1,2)$ - dominating set and it is denoted by $\gamma_{t(1,2)}$(G).

Note that, total $(1,2)$ - dominating set exists only for graphs whose components are of order ≥ 3. Hence, througout this chapter, we consider graphs, whose **components are of order ≥ 3**.

First, we observe that, for any graph G, $\gamma_{t(1,2)}(G) \geq 3$ and $\gamma(G) \leq \gamma_{1,2}(G) \leq \gamma_{t(1,2)}(G)$.

First, we determine the $\gamma_{t(1,2)}$ for standard graphs such as complete graphs, paths, cycles, wheels, fans, stars, complete bipartite graphs, double stars, wounded spiders and spiders.

Next, we study total $(1,2)$ - domination for operations on graphs.

Theorem 1. If G is a connected graph of order $n \geq 3$ and H is any graph, then $\gamma_{t(1,2)}(GoH) = n$.

Theorem 2. Let G be a connected graph of order n_1 and H be a connected graph of order n_2 where $n_1 + n_2 \geq 3$. Then $\gamma_{t(1,2)}(G + H) = 3$.

Next, we derive a bound for $\gamma_{t(1,2)}$ in terms of n and Δ.

Theorem 3. If G is a connected graph with $\Delta(G) \geq n-3$, then $\gamma_{t(1,2)}(G) = 3$.

Theorem 4. Let G be a connected graph with $\Delta(G) \leq n-4$. Then $\gamma_{t(1,2)}(G) \leq n - \Delta(G)$.

Next, we characterize the graphs with $\gamma_{t(1,2)} = n, n-1, n-2$.

Theorem 5. Let G be a connected graph of order $n \geq 3$. Then $\gamma_{t(1,2)}(G) = n \; iff \; n = 3$.

Theorem 6. Let G be a connected graph of order $n \geq 4$. Then $\gamma_{t(1,2)}(G) = n - 1 \; iff \; n = 4$.

Corollary 1. Let G be any graph of order $n \geq 3$. Then

(i) $\gamma_{t(1,2)}(G) = n$ iff $G = sP_3 \cup kK_3$, where $0 \leq s \leq \frac{n}{3}$, $k = \frac{n-3s}{3}$ and $n \equiv 0(mod\ 3)$.

(ii) $\gamma_{t(1,2)}(G) = n - 1$ iff $G = sP_3 \cup kK_3 \cup G_1$, where G_1 is a connected graph of order 4, $0 \leq s \leq \frac{n-4}{3}$, $k = \frac{n-4-3s}{3}$ and $n \equiv 1(mod\ 3)$.

Theorem 7. Let G be a connected graph. Then $\gamma_{t(1,2)}(G) = n-2$ iff one of the following holds: $n = 5$ or $G \in \{C_n, P_n | \ 5 \leq n \leq 8\}$.

Next, we shall characterize the trees with $\gamma_{t(1,2)} = n - \Delta$. We shall prove that the characterization holds for some families of trees. For this purpose, we first introduce notation for these families: $\mathcal{T}_1$ and $\mathcal{T}_2$.

Notation

1. $\mathcal{T}_1$ denotes the family of trees obtained from $K_{1,\ l}, l \geq 3$, where each edge is subdivideed at most 3 times, and at least one edge is subdivided 3 times and at least one edge is subdivided at most 2 times.

2. $\mathcal{T}_2$ denotes the family of trees obtained from $K_{1,l}, l \geq 3$, where each edge is subdivided at most 2 times and at least one edge is subdivided 2 times.

Theorem 8. Let T be a tree with $diam(T) \geq 4$. Then $\gamma_{t(1,2)}(T) = n - \Delta(T)$ iff $T \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \{P_5, P_6, P_7, P_8\}$ or T is a spider or a wounded spider $WS_{s,t}$, where $2 \leq t < s$ and $\mathcal{T}_3$ is the collection of trees in Fig.3.

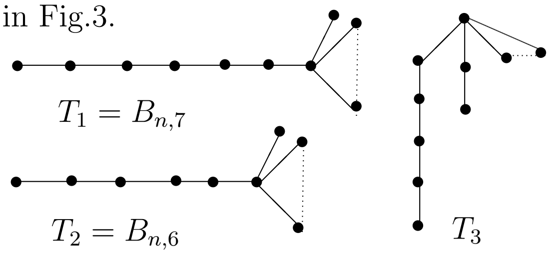


Fig.3.

Next, we derive the bound for $\gamma_{t(1,2)}$ in terms of δ when diameter 2.

Theorem 9. Let G be a graph with $diam(G) = 2$.

If $\delta(G) = 1$, then $\gamma_{t(1,2)}(G) = 3$.

If $\delta(G) \geq 2$, then $\gamma_{t(1,2)}(G) \leq \delta(G) + 1$.

Chapter 6: Bounds of $\gamma_{t(1,2)}$ for Trees and Complement of Graphs

Throughout this chapter r denotes the number of pendant vertices.

In Chapter 5, we have proved that $\gamma_{t(1,2)} \leq n - \Delta$. Now we have derived a stronger bound for trees in terms of the number of pendant vertices.

Theorem 1. Let T be a tree of order $n \geq 3$ having r pendant vertices.

(i) If $r \geq n-3$, then $\gamma_{t(1,2)}(T) = 3$.

(ii) If $2 \leq r \leq n-4$, then $\gamma_{t(1,2)}(T) \leq n-r$. The equality holds when $r = n-4$, $n-5$, $n-6$.

Theorem 2. If T is a tree with diameter 4, then $\gamma_{t(1,2)}(T) = n - r$.

Theorem 3. If T is a tree with diameter 5, then $\gamma_{t(1,2)}(T) = n - r$.

Next, we characterize the trees with $\gamma_{t(1,2)} = 3$.

Theorem 4. Let T be a tree of order ≥ 3. Then $\gamma_{t(1,2)}(T) = 3$ iff T is a star, or a double star, or T is obtained from P_5 by attaching zero or more pendant vertices to the internal vertices of P_5.

Corollary 1. Let G be a connected graph of order ≥ 3. If G' is a spanning subgraph of G, where G' is a star or a double star or G' is a graph obtained from P_5 by attaching zero or more pendant vertices to the internal vertices of P_5, then $\gamma_{t(1,2)}(G) = 3$.

Next, we deal with graph complements.

Theorem 5. For any graph G with $diam(G) \geq 3$, $\gamma_{t(1,2)}(\overline{G}) = 3$.

Theorem 6. Let G be a connected graph of order $n \geq 5$ with $\Delta(G) \leq n-3$. Then $\gamma_{t(1,2)}(\overline{G}) \leq \delta(G) + 3$.

Next, we characterize the connected graphs with $\gamma_{t(1,2)}(\overline{G}) = \delta(G) + 3$.

Theorem 7. Let G be a connected graph of order $n \geq 5$ with $\Delta(G) \leq n-3$. Then $\gamma_{t(1,2)}(\overline{G}) = \delta(G) + 3$ iff $\overline{G} = kK_3 \cup sP_3 \cup H$, where $k, s \geq 0$, $k + s \geq 1$, $|V(H)| = n - 3k - 3s \geq 3$ and H has a full - degree vertex.

Conclusion

In our thesis, we have studied $(1, 2)$ - domination and total $(1, 2)$ - domination.

In Chapter 2, we have obtained $\gamma_{1,2}$ for standard graphs. Also we have characterized the graphs G for which $\gamma_{1,2}(G) = 2, n, n-1$.

In Chapter 3, we have derived upper bounds for $(1, 2)$ - domination number in terms of order, maximum degree and minimum degree. In particular, we have characterized the trees T for which $\gamma_{1,2}(T) = n - \Delta(T)$. Also we have characterized the graphs with $\gamma_{1,2}(G) = n-2, n-3$.

In Chapter 4, we have derived an upper bound of $\gamma_{1,2}$ for trees. Also we have characterized the trees T with $\gamma_{1,2}(T) = 3$. Next we have determined the $(1, 2)$ - domination for graph complements and derived a Nordhaus - Gaddum type bound.

In Chapter 5, we have introduced a new type of domination called total $(1, 2)$ - domination, by imposing a condition on the $(1, 2)$ - dominationg set. First we have determined the total $(1, 2)$ - domination number for standard graphs, composition of graphs and join of graphs. Next, we have derived an upper bound for total $(1, 2)$ - domination number in terms of order and maximum degree. Also we have characterized the trees T for which $\gamma_{t(1,2)}(G) = n, n-1, n-2$.

In Chapter 6, we have derived an upper bound of $\gamma_{t(1,2)}$ for trees. Also we have characterized the trees T with $\gamma_{t(1,2)}(T) = 3$. Next we have determined the total $(1, 2)$ - domination for complement of graphs. Finally we have compared the domination parameters γ, $\gamma_{1,2}$ and $\gamma_{t(1,2)}$.

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