METHODS AND TECHNIQUES TO SOLVE AND ANALYZE THE DIFFERENTIAL EQUATIONS

A Synopsis submitted to Madurai Kamaraj University in partial fulfilment of the requirements for the award of the degree of

DOCTOR OF PHILOSOPHY IN MATHEMATICS

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DECEMBER 2019

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Mathematical model is a tool that allows us to investigate the behaviour of natural / artificial systems. The use of mathematical models in the experimental analysis of behaviour of complex systems has increased over the years, and support in several means to study the system behaviour. Models are also useful for formulating hypotheses and for incorporating new ideas that can later be verified in reality. Models simplify complex problems and are analyzed easily, quickly and inexpensively without actually having to perform the task. Mathematical models provide a common framework that unites diverse behavioral phenomena, systematic approaches to problem solving, to be very specific about objectives and a standardized format for analyzing a problem [21].

Mathematical aspects play an ever increasing role in biology. In recent years mathematical modeling of developmental processes has gained new esteem and has significance in combining diverse areas of knowledge. Not only have mathematical models been used to validate hypotheses made from experimental data, but designing and testing these models have led to testable experimental predictions. There are now impressive cases in which mathematical models have provided fresh insight into biological systems [22].

Mathematical biology aims at the mathematical representation, treatment and modeling of biological processes, using a variety of applied mathematical techniques and tools. Mathematical modeling of interacting species is very challenging as the system is complex as well as multifactorial [25]. It involves different components which have non-linear interaction. Hence, it is also challenging from mathematical point of view to analyze these non-linear models. A Mathematical model used to simulate the three way competition between sponges, macroalgae and coral has been investigated [20]. The community structure of sedentary organisms is largely controlled by the outcome of direct competition for space has been studied [23]. Studying ecology is very important, because a simple change in the environment can have a profound effect on all living things; the destruction of one species can mean the extinction of many others [24]. Due to environmental factors and other related effects the tendency of any species living in a given habitat is to migrate to better suitable regions for its survival and existence. In general, the movement of the species arises due to certain factors such as overcrowding, climate, predator-prey relationship, refuge and fugitive strategies and more importantly due to resource limitations in the given habitat. Mumby et.al [2] introduced a model with grazing at an imposed level demonstrating that a coral reef ecosystem may lose resilience and shift to a coral-depleted state through reductions in grazing intensity. Mumby et.al [3] has extended the role of parrotfish abundance on grazing intensity and implemented management on the system by including control on fishing pressure.

Competition for space between hard corals and macroalgae is fundamental to the overall status of coral reefs because the relative amounts of corals and algae dominant the condition of the reef. Coral reefs are vulnerable to the introduction of seaweeds because they can become ecological dominants, overgrowing corals and displacing native algae. A Mathematical model used to simulate the three-way competition among sponges, macroalgae and coral has been discussed [20]. Most algae reproduce by releasing sexually or asexually produced gametes and/or spores and by vegetative spread and/or fragmentation [13]. The commercial cultivation of the Kappaphycus Algae (KA) in India was stoutly opposed owing to the prediction of its likely invasiveness, as it is exotic to Indian marine environments [6].

A Mathematical model to study the effect of time delay on the transmission dynamics of HIV/AIDS has been analyzed [14]. Understanding the transmission characteristics of infectious disease in communities, regions and countries can lead to better approaches to decreasing the transmission of these disease. An epidemic model describes the transmission process and traces the number of infected population. Such a model can identify the number or proportion of population that left uninfected at the end of an epidemic.

Spatial heterogeneity of mosquito resources, such as hosts and breeding sites, affects mosquito dispersal behavior and consequently affects mosquito population structures, human exposure to vectors and the ability to control disease transmission [5]. By estimating the transmission rate, reproduction number and other variables and parameters, a model can predict whether the associated disease will spread through the population or die out. It can also estimate the impact of a control measure of the disease.

In this thesis multiple aspects of the dynamics of Biolgical species have been studied by Mathematical modeling, Computational analysis and Stability theory.

Homotopy Analysis Method (HAM) is a general analytic approach to get series solutions of various types of nonlinear equations. More importantly, different from all perturbation and traditional non-perturbation methods, the HAM provides us a simple way to ensure the convergence of solution series. It is valid even for strongly nonlinear problems [4,27].

Variational Iteration Method (VIM) is used for analytic treatment of the linear and nonlinear systems of differential equations. The method reduces the calculation size and overcomes the difficulty of handling nonlinear terms. The method shows improvements over existing numerical techniques [15,19].

Homotopy Perturbation Method (HPM) is a very powerful and efficient technique for finding approximate solutions for wide classes of problems. The proper implementation of He's Homotopy Perturbation Method can extremely minimize the size of work if compared to existing traditional techniques [8,18].

The study of equilibrium plays a central role in ordinary differential equations and their applications. An equilibrium point, however, must satisfy a certain stability criterion in order to be significant physically [16]. An equilibrium point of the differential equation is **stable** if all the Eigen values of J, the Jacobian evaluated at that point have negative real parts. The equilibrium point is **unstable** if atleast one of the eigen values has a positive real part. A critical point is **asymptotically stable** if all the eigen values of the Jacobian matrix J are negative or negative real parts. An equilibrium is **locally stable** only if the real part of all eigen values are smaller than zero.

If the behaviour of the dynamical system changes suddenly when a parameter is varied, then it is said to have undergone a **bifurcation** and at a point of bifurcation, stability may be gained or loss [26]. The direction of Hopf bifurcation as well as the stability of the bifurcating periodic orbits by normal form and center manifold theory introduced by Hassard et.al [17]. **Local bifurcations**, which can be analyzed entirely through changes in the local stability properties of equilibria, periodic orbits or other invariant sets as parameters cross through critical thresholds. **Global bifurcations**, which often occur when larger invariant sets of the system collide with each other, or with equilibria of the system.

The local stability of the positive equilibrium and local Hopf bifurcation of a Lotka-Volterra predator-prey model with two delays has been investigated [12]. A Mathematical model of a predator species interacting with prey species is divided into weaker and stronger classes due to predator's catching effciency has been detailed [1]. A stage- structured predator prey model (stage structure on prey) with two discrete time delays has been explained [10]. The effect of the time delay on the stability of the endemically infected equilibrium is investigated [9]. A delayed mathematical model to break the life cycle of Anopheles mosquito has been investigated [11].

Organization of the Thesis

This thesis consists of the following six chapters.

Chapter 1: General Introduction

Chapter 2: Mathematical Preliminaries

Chapter 3: Computational Analysis of Nonlinear Biological Problems

Chapter 4: HIV/AIDS Epidmic Model with an Additional Delay

Chapter 5: Analysis of Delayed Mosquito Life Span Model

Chapter 6: Delayed Model of Removal of Kappapycus Algae in Gulf of Mannar

Chapter 2: Mathematical Preliminaries

We recall some background notions of Computational methods, Delay differential equation models, HIV model, Mosquito population model and Coral reef model assumed in the rest of the theory.

Chapter 3: Computational Analysis of Nonlinear Biological Problems

A Mathematical model has been developed to simulate the three-way competition among sponges, macroalgae and coral with grazing term [20]:

$$\frac{du}{dt} = ru\left(1-u-v-w\right) - a_1uv - a_2uw \tag{1}$$

$$\frac{dv}{dt} = a_1uv + a_3vw + mv\left(1-u-v-w\right) - \frac{gv}{v+\left(1-u-v-w\right)+bw} \tag{2}$$

$$\frac{dw}{dt} = a_2uw + zw\left(1-u-v-w\right) - a_3vw - \frac{gbw}{v+\left(1-u-v-w\right)+bw} \tag{3}$$

and without grazing term (i.e., g = 0)

$$\frac{du}{dt} = ru\left(1-u-v-w\right) - a_1uv - a_2uw \tag{4}$$

$$\frac{dv}{dt} = a_1uv + a_3vw + mv\left(1-u-v-w\right) \tag{5}$$

$$\frac{dw}{dt} = a_2uw + zw\left(1-u-v-w\right) - a_3vw \tag{6}$$

where r - coral growth rate,

a_1 - the rate at which macroalgae overgrow coral,

a_2 - the rate at which sponge overgrow coral,

z - the rate at which sponge overgrow turf,

a_3 - the rate at which macroalgae overgrow sponge,

m - the rate at which macroalgae overgrow turf

g - the grazing rate

b - grazing preference on sponges.

Model equations are solved by Homotopy Perturbation Method (HPM), Variational Iteration Method (VIM) and Homotopy Analysis Method (HAM), when the presence of the grazing term equations (1-3) and absence of the grazing term equations (4-6). The Matlab output shows that the growth rate of v (macroalgae) in the absence of the grazing term increases than in the presence of the grazing term during the growth rates of u (coral) and w (sponges) sustain the same level. The growth rate of v (macro-algae) increases during the absence of the grazing term than the presence of the grazing term. The growth rate of u (coral) and w (sponges) show almost the same level in both the presence and the absence of the grazing term.

Chapter 4: HIV/AIDS Epidemic Model with an Additional Delay

A HIV/AIDS epidemic model with screening and time delays has been discussed [14]. We have incorporated a time delay between the time of contact of an infected person and the time of known into the model, we get:

$$\frac{dS}{dt} = Q_0 - (\beta_1 I_1 + \beta_2 I_2(t-\tau_2))S - \mu S \tag{7}$$

$$\frac{dI_1}{dt} = (\beta_1 I_1 + \beta_2 I_2(t-\tau_2))S - (\mu+\delta)I_1 - \theta I_1(t-\tau_1) \tag{8}$$

$$\frac{dI_2}{dt} = \theta I_1(t-\tau_1) - (\mu+\delta)I_2 \tag{9}$$

where S - susceptible population,

I_1 - the infectives that do not know they are infected,

I_2 - the infectives that know they are infected,

Q_0 - constant rate of immigration of susceptibles,

β_i(i=1,2) are the per capita contact rates for susceptibles individuals with (unaware, aware) infectives respectively,

μ - the natural mortality rate unrelated to AIDS,

θ - the rate of unaware infectives to become aware infectives by screening,

δ - the rate by which both types of infectives develop AIDS,

τ_1 - the time from screening of HIV until the result of screening become available,

τ_2 - the time of contact of an infected person and the time of known.

We have analyzed a mathematical model to study the effect of time delays. The stability of the disease-free equilibrium and the endemic equilibrium on the introduction of an additional time delay have been investigated. Also, the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits has been discussed. Finally, numerical simulations are carried out towards the theoretical predictions to optimize the system of delay differential equations (7)-(9).

Chapter 5: Delayed Mosquito Life Span Model

A mosquito population dynamics model without movement has been developed [5]. We have incorporated the time delay from mosquitoes searching for oviposition sites into searching for hosts into the nonlinear system of equations [5]. The delay model is given by,

$$E' = b\rho_{A_o}A_o - (\mu_E + \rho_E)E \quad (10)$$

$$L' = \rho_E E - (\mu_{L_1} + \mu_{L_2}L + \rho_L)L \quad (11)$$

$$P' = \rho_L L - (\mu_P + \rho_P)P \quad (12)$$

$${A_h}' = \rho_P P + \rho_{A_o}A_o(t-\tau) - (\mu_{A_h} + \rho_{A_h})A_h \quad (13)$$

$${A_r}' = \rho_{A_h}A_h - (\mu_{A_r} + \rho_{A_r})A_r \quad (14)$$

$${A_o}' = \rho_{A_r}A_r - \mu_{A_o}A_o - \rho_{A_o}A_o(t-\tau) \quad (15)$$

where b - number of female eggs laid per oviposition.

ρ_E - egg hatching rate into larvae (day^{-1}).

ρ_L - rate at which larvae developing into pupae (day^{-1}).

ρ_P - rate at which pupae develop into adult or emergence rate (day^{-1}).

μ_E - egg mortality rate (day^{-1}).

μ_P - pupae mortality rate (day^{-1}).

μ_{L_1} - density independent larvae mortality rate (day^{-1}).

μ_{L_2} - density dependent larvae mortality rate (day^{-1}).

ρ_{A_h} - rate at which host-seeking mosquitoes enter the resting state (day^{-1}).

ρ_{A_r} - rate at which resting mosquitoes enter oviposition site searching state (day^{-1}).

ρ_{A_o} - oviposition rate (day^{-1}).

μ_{A_h} - mortality rate of mosquitoes of searching for hosts (day^{-1}).

μ_{A_r} - mortality rate of resting mosquitoes (day^{-1}).

μ_{A_o} - mortality rate of mosquito searching for oviposition sites (day^{-1}).

The effect of delay on the stability of the persistent equilibrium and Hopf bifurcation has been investigated. Finally, numerical simulations have been executed.

Chapter 6: Delayed Model of Removal of Kappaphycus Algae in Gulf of Mannar

Mathematical model has been developed to analyze the stability of the spatial competition among coral, Kappaphycus Algae (KA) and Native Algae (NA). Chandrasekaran et.al [6] reported the shifting of algal dominated reef ecosystem due to the invasion of KA in Gulf of Mannar. Subsequently, the dominance of KA over NA and corals in competing for space has also been reported. To simulate the three way competition among corals, KA and NA, we have proposed a model as a system of non-linear ordinary differential equations. The manual removal of KA from the coral reefs has been reported, we have incorporated the manual removal rate term d of KA in the proposed model.

KA sexual reproduction by spores in the Gulf of Mannar Marine Bio-sphere Reserve (GoM) in future, when environmental conditions unanimously favor this algae has been deliberated [7]. NA can produce sexually and asexually by forming flagellate and sometimes non-flagellate spores. The vegetative propagation is achieved through fragmentation has been reported [13]. To analyze the impact of delays in propagation of KA and NA through vegetation/spores on coral-algae interactions, we have proposed the following delay model:

$$\frac{dx}{dt} = rx - rx^2 - rxy - rxz - a_1xy - a_2xz + dy \tag{16}$$

$$\frac{dy}{dt} = a_1yx + a_3yz + \nu y(t-\tau_1) - \nu yx - \nu y^2 - \nu yz - dy \tag{17}$$

$$\frac{dz}{dt} = a_2zx + hz(t-\tau_2) - hxz - hzy - hz^2 - a_3zy \tag{18}$$

where x - Coral growth with respect to time (t),

y - KA growth with respect to time (t),

z - NA growth with respect to time (t),

1-$(x+y+z)$ - Turf growth with respect to time (t),

d - the manual removal rate of KA.

r - coral growth on turf $(cm^2/cm^2/15d)$,

a_1 - KA growth on coral $(cm^2/cm^2/15d)$,

a_2 - NA growth on coral $(cm^2/cm^2/15d)$,

a_3 - KA growth on NA $(cm^2/cm^2/15d)$,

ν - KA growth on turf $(cm^2/cm^2/15d)$,

h - NA growth on turf $(cm^2/cm^2/15d)$.

τ_1 - time delay of propagation of KA through vegetation/spores,

τ_2 - time delay of propagation of NA through vegetation/spores.

We have analyzed the effect of two time delays on the coral reef system with the manual removal rate of KA. The actual manual removal rate of KA is very much lesser than our vision in the field. For non-delay case the growth rate of coral increases after the manual removal rate of KA. Then, we have considered the delay case. In the presence of the time delay of propagation of KA through vegetation/spores, the growth rate of KA increases. In the absence of the time delay of propagation of KA through vegetation/spores, the time evolution of three populations are asymptotiocally stable. In the presence of both the time delays (τ_1, τ_2), the system is asymptotically stable. Finally, the impact of manual removal rate of KA triggers its growth. Also, we have derived the bifurcating periodic solutions are orbitally asymptotically stable with an asymptotical phase if $\beta_2 < 0$ and unstable if $\beta_2 > 0$ and the period of bifurcating periodic solutions increases if $T_2 > 0$ and decreases if $T_2 < 0$.

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